Math 4200
Monday November 30
Chapter 5: 5.1-5.2 conformal maps and fractional linear transformations, continued. A geometric way to understand FLT's in terms of the Riemann sphere. (movie!) Intro to Riemann surfaces, leading into Daniel's presentation.

Announcements: I changed the due date for last week's homework 13 to be tonight at midnight.

Math 4200-001
Week 14 concepts and homework
5.1-5.2
Due Friday December 4 at 11:59 p.m.

This final homework assignment is optional - if you do it, you can choose to use it for 30% of your final exam score. Unlike on our midterms and on the final exam itself, I encourage you to collaborate on this assignment as if it was a regular homework assignment.

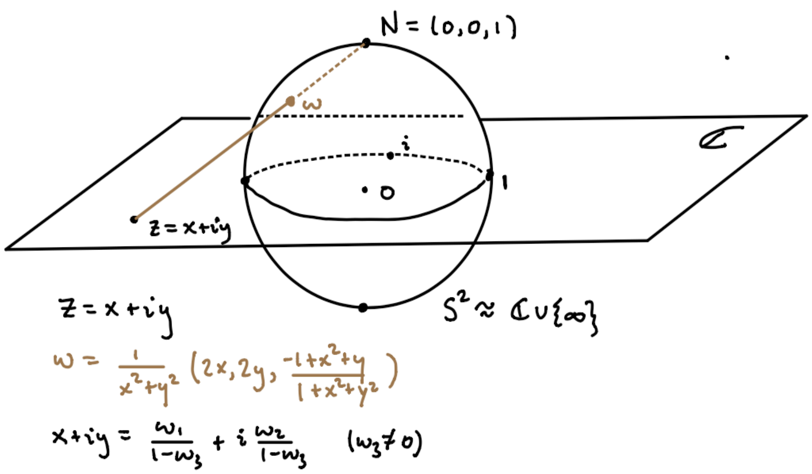
5.1: 10, 11, 12.
5.2 1, 4a, 6, 7, 9, 10, 17, 24, 26.

w14.1 Use the result of 5.1.12 to show that the only conformal bijections of the Riemann sphere are given by the fractional linear transformations. Hint: If a conformal bijection of the Riemann sphere maps ∞ to ∞ then restricting it to $\mathbb{C}$ yields a conformal bijection of $\mathbb{C}$ to itself.

Remember that on Friday we checked that FLTs map the set of circles and lines to itself. Also, they are bijections of the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

$$f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

There are interesting ways to visualize FLT's on the Riemann sphere. Recall, we talked about the identification of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the unit sphere $S^2 \subseteq \mathbb{R}^3$ when we were discussing the classification of isolated singularities and the reason for the word "pole" when describing the case of Laurent series which have a finite number of negative powers. Below is the picture I drew at the time, of stereographic projection from the unit sphere $S^2 \subseteq \mathbb{R}^3$ to the $x - y$ coordinate plane identified with $\mathbb{C}$, and its inverse transformation. One can show algebraically that circles on the sphere correspond to circles and lines in $\mathbb{C}$ under stereographic projection. In fact, if the sphere circle goes through the north pole, then the image in $\mathbb{C}$ is a straight line. Otherwise the sphere circle corresponds to a $\mathbb{C}$-circle!



In this way one can identify FLTs as being magically related to the natural Euclidean motions of a sphere $\mathbb{R}^3$ - namely translations and rotations and their compositions, combined with various stereographic projections from those displaced spheres. Here's a fun short video in that vein

https://www.youtube.com/watch?v=0z1fIsUNhO4

Using FLTs to construct various conformal transformation: Notice that

$$f(z) = \frac{z-a}{z-b}\left(\frac{c-b}{c-a}\right)$$

maps

$$a \to 0$$
$$b \to \infty$$
$$c \to 1.$$

Since 3 points uniquely determine particular circles and lines one can use FLT's to map any circle or line to any other circle or line.

Using functions of this form, and their inverses, one can construct FLT's to map triples of points to triples of points:

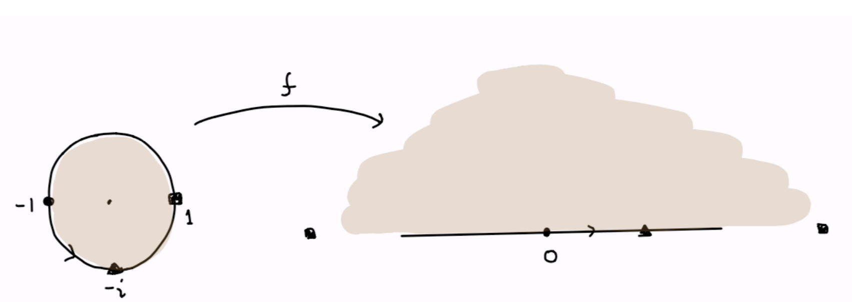
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Thus you can map any line or circle to any other line or circle. If you're trying to build a conformal map from one domain to another and parts of the domains are bounded by circular arcs, lines or rays, FLT's may be building blocks in your construction.

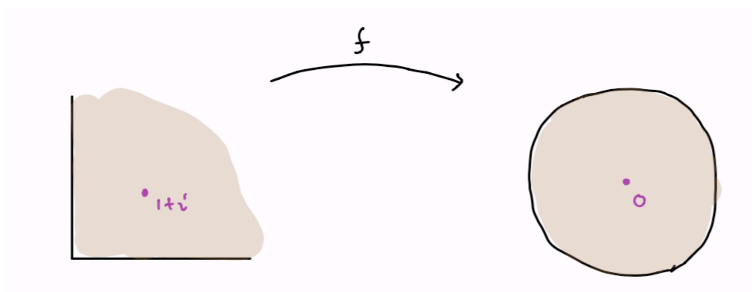
Example Find a FLT from the unit disk to the upper half plane by mapping

$$-1 \to 0$$
$$1 \to \infty$$
$$-i \to 1$$

and making any necessary adjustments. (By magic, once you know the boundary of the disk goes to the real axis, you only have to check that one interior point goes to an interior point, or that the orientation is correct along the boundary, to know that you're mapping the unit disk to the upper half plane instead of the lower half plane. The proof of the magic theorem is an appendix in today's notes.)



Example Find a conformal transformation of the first quadrant to the unit disk, so that the image of $1 + i$ is the origin. How many such conformal transformations are there? It's fine to write your transformation as a composition.



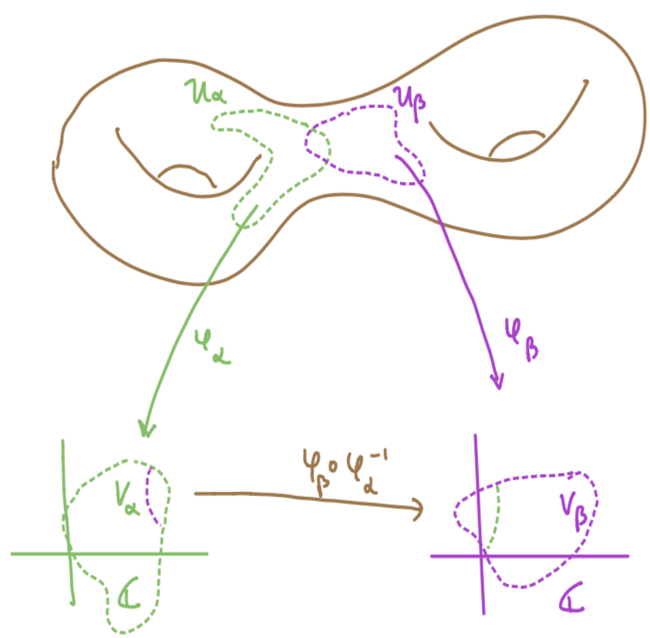
More on the Riemann sphere and general Riemann surfaces:

Definition A *Riemann surface* S is a connected topological space S together with an *atlas* consisting of *charts* $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ where the following three properties hold

(1) $\bigcup_{\alpha \in A} U_\alpha = S$ and each U_α is open and connected.

(2) Each $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}$ is a *homeomorphism.* We can call the individual pairs $\left(V_\alpha, \varphi_\alpha^{-1}\right)$ the *pages* of the atlas.

(3) The *transition maps* between parts of the pages of the *atlas* $\varphi_\beta \circ \varphi_\alpha^{-1} : \ \varphi_\alpha\left(U_\alpha \cap U_\beta\right) \to \varphi_\beta\left(U_\alpha \cap U_\beta\right)$ are all conformal.



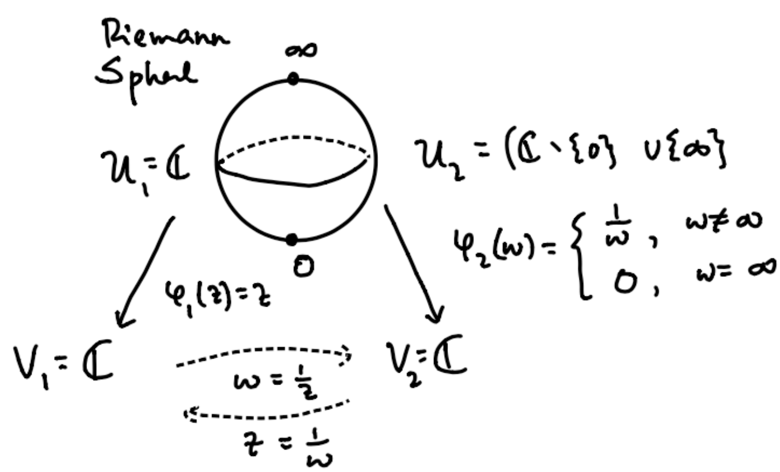
This definition makes sense when you think of what an actual *geographical* atlas is. Here are a couple examples:

- The complex plane itself, or any connected open set in the complex plane is a Riemann surface which has one possible atlas consisting of a single page, with $U = V$ and $\varphi = id$.

- The Riemann sphere $\mathbb{C} \cup \{\infty\}$, which is homeomorphic to $S^2 \subseteq \mathbb{R}^3$, as we've discussed. The easiest atlas to use has just two pages. One page describes everything except the origin. The other one describes everything except ∞.

$$U_1 = \mathbb{C},\ \varphi_1 : U_1 \to V_1 = \mathbb{C},$$
$$\varphi_1(z) = z$$
$$U_2 = (\mathbb{C} \backslash \{0\}) \cup \{\infty\},\ \varphi_2 : U_2 \to V_2 = \mathbb{C}$$
$$\varphi_2(w) = \begin{cases} \dfrac{1}{w} & w \neq \infty \\ 0 & w = \infty \end{cases}$$

Then $U_1 \cap U_2$ is the punctured complex plane $\mathbb{C} \setminus \{0\}$ and
$\varphi_2 \circ \varphi_1^{-1}(z) = \frac{1}{z};\ \varphi_1 \circ \varphi_2^{-1}(w) = \frac{1}{w}$.



Definition: Let S_1, S_2 be Riemann surfaces, and $f: S_1 \to S_2$ be a function. Then f is *holomorphic* (or analytic) if and only if each of the corresponding maps from atlas pages of S_1 to atlas pages of S_2 are analytic. Precisely, given an atlas for S_1:

$$\left\{U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha\right\}_{\alpha \in A}$$

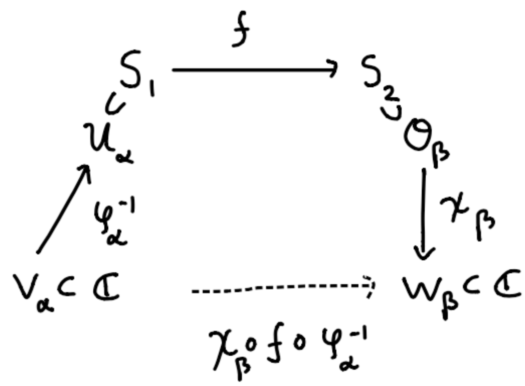
and and atlas for S_2

$$\left\{O_\beta, \chi_\beta : O_\beta \to W_\beta\right\}_{\beta \in B}$$

then f is defined to be holomorphic if and only if each triple composition

$$\chi_\beta \circ f \circ \varphi_\alpha^{-1} : V_\alpha \to W_\beta$$

is analytic.



So for example, for a function $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ there are two cases to consider, in order to deduce whether f is analytic near z_0, as a map of Riemann surfaces:

$f(z_0) \in \mathbb{C}$: usual definition.

$f(z_0) = \infty$: Does $\frac{1}{f(z)}$ have a removable singularity at z_0? In other words does $f(z)$ have a pole at z_0, so that $f(z_0) = \infty$?

The text defined a *meromorphic function* on $\mathbb{C}$ to be one which is analytic except for a countable number of isolated pole singularities. This is the same as saying that $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is holomorphic as a function between Riemann surfaces.

More generally, a function $f: S \to \mathbb{C}$ from a Riemann surface to $\mathbb{C}$ with isolated singularities is said to be *meromorphic* if and only if the function f extends to be holomorphic as a function $f: S \to \mathbb{C} \cup \{\infty\}$.

Warmup for Daniel's presentation: When we have a Riemann surface it's important to understand the analytic and meromorphic functions on that surface. For example, entire functions on $\mathbb{C}$ are special, as are meromorphic functions on $\mathbb{C}$.

By the Riemann mapping theorem, the spaces of analytic and meromorphic functions on any simply connected domain except $\mathbb{C}$ can be identified with the space of analytic and meromorphic functions on the unit disk. What about on domains with one or more holes? How about domains which are other Riemann surfaces?

<u>Theorem</u>: Let $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C}$ be analytic. Then f is constant. More generally, let S be any (connected) compact Riemann surface. Then the only analytic functions $f: S \to \mathbb{C}$ are constants.

proof: analytic functions $f(z)$ are continuous, so if the domain surface is compact then then $|f(z)|$ attains its maximum value, say at $P \in S$. Pick an atlas page for P, i.e.

$$P \in U_\alpha, \ \varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}.$$

Then $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{C}$ is analytic and $|f \circ \varphi_\alpha^{-1}|$ has an interior maximum value at $\varphi_\alpha(P)$ so is constant on V_α by the maximum principle. Then argue that the set where f equals this constant is both open and closed!

<u>Theorem</u>: The only meromorphic functions on the Riemann sphere are rational functions!

proof: Because the Riemann sphere is compact the number N of isolated singularities is finite. First assume that ∞ is not a singular point. Then at each singularity $z_k \in \mathbb{C}$ let

$$S_k(z) = \sum_{m=1}^{M_m} \frac{b_{km}}{(z - z_k)^m}$$

be the singular part of the Laurent series. Consider the difference

$$g(z) = f(z) - \sum_{k=1}^{N} S_k(z).$$

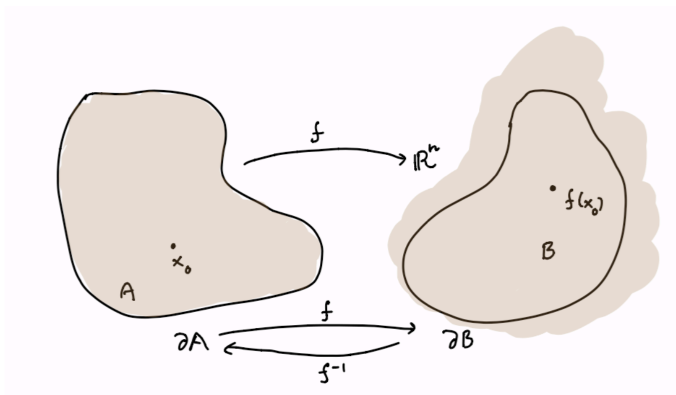
Then $g(z)$ is bounded at infinity, because $f(z)$ is and the $S_k(z)$ converge to zero there. Notice that $g(z)$ has removable singularities at the original singular points. Thus $g(z)$ extends to be entire and bounded, hence constant. Thus $f(z)$ is a rational function.

In the case that ∞ is a singular point for f, then that means the Laurent series for $f\left(\frac{1}{z}\right)$ at $z = 0$ has a finite number of negative powers, and so the "singular" Laurent series for the original f near infinity consists of a polynomial, and the regular part at infinity is the portion of that Laurent series with negative powers. Repeat the argument as above, except also subtract off this polynomial to get the bounded analytic function on $\mathbb{C}$ which must be constant.

Appendix: Magic Theorem Let $A, B \subseteq \mathbb{R}^n$ be open, connected, bounded sets. Let $f: A \to \mathbb{R}^n$, $f \in C^1$, with $df_x : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ invertible $\forall\, x \in A$ (i.e. the Jacobian matrix is invertible). Furthermore, assume

- $f: \overline{A} \to \mathbb{R}^n$ is continuous and one-to-one.
- $f(\delta A) = \delta B$
- $f(x_0) \in B$ for at least one $x_0 \in A$.

Then $f(A) = B$ and f is a global *diffeomorphism* between A and B. (i.e. $f^{-1}: B \to A$ is also differentiable), and $f^{-1}: \overline{B} \to \overline{A}$ is continuous.



proof: Step 1: $f(A) \subseteq B$.

proof: Let

$$O := \{x \in A \mid f(x) \in B\}$$

Then

- $x_0 \in O$
- O is open by the local inverse function theorem, since $x_1 \in O$ and $f(x_1) \in B$ implies there is a local inverse function from an open neighborhood of $f(x_1)$ in B, back to a neighborhood of x_1 in A.
- O is closed in A because if $\{x_k\} \subseteq O$, $\{x_k\} \to x \in A$ then $\{f(x_k)\} \to f(x)$ and since $\{f(x_k)\} \subseteq B$ we have $f(x) \in \overline{B}$. But since f is one-one and maps the boundary of A bijectively to the boundary of B, $f(x)$ cannot be in the boundary of B. Thus $f(x) \in B$.
- Thus, since A is connected, O is all of A, and $f(A) \subseteq B$.

Step 2: $f(A) = B$.

proof:

- $f(A)$ is open (by the local inverse function theorem again), so $f(A) \subseteq B$ is open.
- And $f(A)$ is closed in B because if

$$\{f(x_k)\} = \{y_k\} \subseteq f(A), \text{ with } \{y_k\} \to y \in B,$$

then because $\overline{A}$ is compact, a subsequence $\{x_{k_j}\} \to x \in \overline{A}$ with $\{f(x_{k_j})\} \to f(x) = y$, so $x \notin \delta A$ because $y \in B$, so $x \in A$ and $y \in f(A)$.

- So, because B is connected, $f(A)$ is all of B.

QED.

Remark: In $\mathbb{C}$ you can also imply this theorem to unbounded domains, i.e. in $\mathbb{C} \cup \{\infty\}$ because of the following diagram, in which $f_2 \circ f \circ f_1^{-1}$ satisfies the hypotheses of the original theorem: